ON INTEGRAL INEQUALITIES SIMILAR TO QI'S INEQUALITY

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ABSTRACT

In this paper, we give some generalization of the results contained in Pecaric, Ngo et.al. and Pejkovic. These were done by introducing n-terms of functions for all natural number $n \in N$ on a multiple QI's integral inequality.

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INTRODUCTION

Qi (2002) obtained the following new integral inequality:
Suppose $n \geq 1$ be an integer and suppose that f has a continuous derivative of the n-th order on [a,b], $f'(a) \geq 0$ and $f^{(n)} \geq n!$ where $0 \leq i \leq n-1$. Then

$$\int_a^b [f(x)]^{n+2}\, dx \geq \left[\int_a^b f(x)\, dx\right]^{n+1} \qquad (1)$$

He then proposed the following problem:

Under what conditions does the inequality

$$\int_a^b [f(x)]^t\, dt \geq \left(\int_a^b f(x)\, dx\right)^{t-1} \qquad (2)$$

hold for $t > 1$?

Towghi (2001) found sufficient condition for (2) to hold. To recall the result of Towghi, we need some notations. Let $f^{(0)} = f$, $f^{(-1)} = \int_a^x f(s)\, ds$, and $[x]$ denote the greatest integer less than or equal to x. For $t \in (n, n+1]$, where n is a positive integer, let $\Gamma(t) = t(t-1)(t-2)...(t-(n-1))$. For $t < 1$, let $\Gamma(t) = 1$ and $t > 1$, $x \in [a,b]$, and $f^{(i)}(a) \geq 0$ for $0 \leq i \leq [t-2]$. If $f^{([t-2])}(x) \geq \Gamma(t-1)(x-a)^{t-[t]}$, then $(b-a)^{t-1} \leq \int_a^b f(x)\, dx$, and (2) holds.

Since $n \geq 1$ is an integer and suppose that f satisfies the conditions of the above problem. Then, from $f^{(n)}(x) \geq n!$ and $f^{(i)}(a) \geq 0$ for $0 \leq i \leq m-1$, it follows that $f^{(i)}(t) \geq 0$ and are non-decreasing for $0 \leq i \leq n-1$. In particular, f is nonnegative when $t \geq 2$, the assumption above also imply that f is nonnegative. The proof of (2) in this case is made by the use of the integral version of Jensen's inequality.

By using a lemma of convexity and Jensen's inequality, Yu and Feng (2001) established the following result: Suppose that f is a continuous function on $[a,b]$ satisfying the following condition:

$$\int_a^b f(x)\, dx \geq (b-a)^{t-1} \qquad (3)$$

Then, we have

$$\int_a^b [f(x)]^t \, dx \geq \left[\int_a^b f(x)\, dx\right]^{t-1} \qquad (4)$$

Pogany (2002) found sufficient conditions for more general inequality

$$\int_a^b [f(x)]^\alpha \, dx \geq \left[\int_a^b f(x)\, dx\right]^{\beta} \qquad (5)$$

to hold without assuming the differentiability on the function f and without using convexity criteria. Pogany established some inequalities which are generalization, reversed form, or weighted version of (2). Mazouzi and Feng (2003) established a functional inequality from which the inequality (2) and other integral or discrete inequalities can be deduced.

In response to the open problem (2), an affirmative answers, extensions, reversed forms, and interpretations of inequality (2) can be found in Csiszar and Mori (2003) and the references therein.

Qi and Yu (2001) first gave an affirmative answer to this open problem using the integral version of Jensen's inequality and a lemma of convexity. The second affirmative answer to (2) was given by Towghi (2001). Pogany (2002) was motivated by (2) and proved the following:

$$\int_a^b [f(x)]^\alpha \, dx \geq \left(\int_a^b f(x)\, dx\right)^{\beta} \qquad (6)$$

and it's reversed form under assumptions of the bounds, depending on $b-a$, α and β, and convexity of the function f, which contains an answer to the above open problem and some reversed forms of (2).

Mazouzi and Qi (2003) employed a functional inequality which is an abstract generalization of the classical Jensen's inequality and functional inequality (8) was established, from which inequality (2), some integral inequalities and an interesting discrete inequality involving sums can be deduced.

Let ℓ be a linear vector space of real-valued functions, p and q be two real numbers such that $p \geq q \geq 1$. Assume that f and g are two positive functions in ℓ and G is a positive linear form on ℓ such that $G(g) > 0$, $fg \in \ell$, and $gf^p \in \ell$. If

$$[G(gf)]^{p-q} \geq [G(g)]^{p-1} \qquad (7)$$

then

$$G(gf^p) \geq [G(gf)]^q \qquad (8)$$

Very recently, Csiszar and Mori (2003) interpreted inequality (6) in terms of moments as:

$$E(X^\alpha) \geq C(EX)^\beta \qquad (9)$$

where $C = (b-a)^{\beta-1}$ and $X = f$ is a random variable. To demonstrate the power of the convexity in probability theory, among other things, they found sharp conditions on the range of X, under which (9) or the converse inequality holds for fixed $0<\beta<\alpha$. Hence, the results by Pogany (2002) were improved slightly by a factor of at least $\left(\frac{3}{2}\right)^{\frac{\alpha}{\alpha-\beta}}$.

Similar to results in (2), Feng propose the following problem:

Problem 2 : Under what conditions does the inequality:

$$\int_a^b [f(x)]^t\,dx \le \left(\int_a^b f(x)dx\right)^{1-t} \qquad (10)$$

hold for $t<1$?

In this paper, by utilizing the reversed Holder's inequality in Liu (1990) and a reversed convolution inequality in Sai Toh, Tun and Yamamoto (2003), we establish some new Qi type integral inequalities which extend and generalized some earlier results in literature.

MATERIAL AND METHOD: INTEGRAL INEQUALITY SIMILAR TO QI'S INEQUALITY

We first state the QI's inequality and also discuss some propositions on it.

Problem 1 :

Under what conditions does the inequality

$$\int_a^b [f(x)]^t\,dx \ge \left(\int_a^b f(x)dx\right)^{t-1}$$

hold for $t>1$?

Similar to problem 1, we propose the following :

Problem 2 [Feng (2002)]:

Under what conditions does the inequality:

$$\int_a^b [f(x)]^t\,dx \le \left(\int_a^b f(x)dx\right)^{1-t} \qquad (11)$$

hold for $t<1$?

Before giving an affirmative answer to problem 2, the following propositions are established:

Proposition 1 :

Let f and g be non-negative functions with $0<m\le\frac{f(x)}{g(x)}\le M<\infty$ on [a,b]. Then for $p>1$ and $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ we have

$$\int_a^b [f(x)]^{\frac{1}{p}}[g(x)]^{\frac{1}{q}}\,dx \le M^{\frac{1}{p^2}} m^{-\frac{1}{q^2}} \int_a^b [f(x)]^{\frac{1}{q}}[g(x)]^{\frac{1}{p}}\,dx \qquad (12)$$

then,

$$\int_a^b [f(x)]^{\frac{1}{p}}[g(x)]^{\frac{1}{q}}\,dx \le M^{\frac{1}{p^2}} m^{-\frac{1}{q^2}} \left(\int_a^b [f(x)]dx\right)^{\frac{1}{q}}\left(\int_a^b [g(x)]dx\right)^{\frac{1}{p}} \qquad (13)$$

The following gives an affirmative answer to problem 2 as follows:

Proposition 2:

For a given positive integer $p \geq 2$, if $0 < m \leq f(x) \leq M$ on $[a,b]$ with $M \leq m^{(p-1)^2}/(b-a)^p$, then

$$\int_a^b [f(x)]^{\frac{1}{p}} dx \leq \left(\int_a^b [f(x)] dx\right)^{1-\frac{1}{p}} \qquad (14)$$

Remark:

Now we discuss a simple case of "equality" in proposition 2. If we make the substitution $f(x) = M = m$ and $b - a = 1$ with $p = 2$, then the following in (14) holds, that is

$$\int_a^b [f(x)]^{\frac{1}{p}} dx \leq \left(\int_a^b [f(x)]\, dx\right)^{1-\frac{1}{p}}$$

$$\Rightarrow \int_a^b M^{\frac{1}{2}} dx \leq \left(\int_a^b M\ dx\right)^{1-\frac{1}{2}}$$

$$= \left(\int_a^b M\ dx\right)^{\frac{1}{2}} = \int_a^b M^{\frac{1}{2}} dx$$

In order to illustrate a possible practical use of proposition 2, we shall give in the following simple example is a direct application of inequality (14).

1. **Example**

(1) Let $f(x) = 8x^2$ on $\left[\frac{1}{2}, 1\right]$ with $M = 8$ and $m = 2$. Taking $p = 2$, we see that the conditions of proposition 2 are fulfilled and straight forward computation yields:

Solution:

$$\int_a^b [f(x)]^{\frac{1}{p}} dx < \left(\int_a^b [f(x)]\, dx\right)^{1-\frac{1}{p}}$$

$$\Rightarrow \int_{\frac{1}{2}}^1 \left(8x^2\right)^{\frac{1}{2}} dx < \left(\int_{\frac{1}{2}}^1 8x^2\ dx\right)^{1-\frac{1}{2}}$$

from the left hand side, we have

$$\int_{\frac{1}{2}}^1 \left(8x^2\right)^{\frac{1}{2}} dx = \int_{\frac{1}{2}}^1 8^{\frac{1}{2}} x\, dx = \left.\frac{8^{\frac{1}{2}} x^2}{2}\right|_{\frac{1}{2}}^1$$

$$= \frac{3}{4}\sqrt{2}$$

Also, from the right hand side, it implies

$$\left(\int_{\frac{1}{2}}^{1} 8x^2\,dx\right)^{\frac{1}{2}} = \left(\frac{8x^3}{3}\Bigg|_{\frac{1}{2}}^{1}\right)^{\frac{1}{2}}$$

$$\Rightarrow \left(\frac{8}{3}-\frac{1}{3}\right)^{\frac{1}{2}} = \left(\frac{7}{3}\right)^{\frac{1}{2}}$$

Therefore,

$$\frac{3}{4}\sqrt{2} < \frac{\sqrt{7}}{\sqrt{3}}$$

GENERALIZATION OF QI'S INTEGRAL INEQUALITY

The objective of this section is to obtain the n-dimensional functions of the QI's integral inequality.

Theorem 1:

Let $\prod_{i=1}^{n} f_i(x)$ and $\prod_{i=1}^{n} g_i(x)$ be non-negative functions with $0 < m \le \dfrac{\prod_{i=1}^{n} f_i(x)}{\prod_{i=1}^{n} g_i(x)} \le M < \infty$ on [a,b]. Then, for $p > 1$ and $q > 1$ with $\dfrac{1}{p}+\dfrac{1}{q}=1$ we have

$$\int_a^b [f_1(x).f_2(x)..., f_n(x)]^{\frac{1}{p}} [g_1(x).g_2(x)...g_n(x)]^{\frac{1}{q}}\,dx \le M^{\frac{1}{p^2}} m^{-\frac{1}{q^2}} \int_a^b [f_1(x).f_2(x)$$

$$...f_n(x)]^{\frac{1}{q}} [g_1(x).g_2(x)...g_n(x)]^{\frac{1}{p}}\,dx \qquad (15)$$

$$\int_a^b \left[\prod_{i=1}^{n} f_i(x)\right]^{\frac{1}{p}} \left[\prod_{i=1}^{n} g_i(x)\right]^{\frac{1}{q}} dx \le M^{\frac{1}{p^2}} m^{-\frac{1}{q^2}} \left(\int_a^b \prod_{i=1}^{n} f_i(x)dx\right)^{\frac{1}{q}} \left(\int_a^b \prod_{i=1}^{n} g_i(x)dx\right)^{\frac{1}{p}} \qquad (16)$$

Proof :

Using Holder's Inequality on the left hand side of (22) implying

$$\int_a^b \left[\prod_{i=1}^{n} f_i(x)\right]^{\frac{1}{p}} \left[\prod_{i=1}^{n} g_i(x)\right]^{\frac{1}{q}} dx \le \left(\int_a^b \prod_{i=1}^{n} f_i(x)dx\right)^{\frac{1}{p}} \left(\int_a^b \prod_{i=1}^{n} g_i(x)dx\right)^{\frac{1}{q}} \qquad (17)$$

$$= \left(\int_a^b \left[\prod_{i=1}^{n} f_i(x)\right]^{\frac{1}{p}} \left[\prod_{i=1}^{n} f_i(x)\right]^{\frac{1}{q}} dx\right)^{\frac{1}{p}} \left(\int_a^b \left[\prod_{i=1}^{n} g_i(x)\right]^{\frac{1}{p}} \left[\prod_{i=1}^{n} g_i(x)\right]^{\frac{1}{q}} dx\right)^{\frac{1}{q}} \qquad (18)$$

Since, $\left[\prod_{i=1}^{n} f_i(x)\right]^{\frac{1}{p}} \le M^{\frac{1}{p}}\left[\prod_{i=1}^{n} g_i(x)\right]^{\frac{1}{p}}$ and $\left[\prod_{i=1}^{n} g_i(x)\right]^{\frac{1}{q}} \le m^{-\frac{1}{q}}\left[\prod_{i=1}^{n} f_i(x)\right]^{\frac{1}{q}}$, from the given inequality $0 < m \le \dfrac{\prod_{i=1}^{n} f_i(x)}{\prod_{i=1}^{n} g_i(x)} \le M < \infty$

we have,

$$\int_a^b \left[\prod_{i=1}^{n} f_i(x)\right]^{\frac{1}{p}} \left[\prod_{i=1}^{n} g_i(x)\right]^{\frac{1}{q}} dx = \left(\int_a^b \left[\prod_{i=1}^{n} f_i(x)\right]^{\frac{1}{p}} dx\right)^{\frac{1}{p}} \left(\int_a^b \left[\prod_{i=1}^{n} f_i(x)\right]^{\frac{1}{q}} dx\right)^{\frac{1}{p}}$$

$$\times \left(\int_a^b \left[\prod_{i=1}^{n} g_i(x)\right]^{\frac{1}{p}} dx\right)^{\frac{1}{q}} \left(\int_a^b \left[\prod_{i=1}^{n} g_i(x)\right]^{\frac{1}{q}} dx\right)^{\frac{1}{q}}$$

$$\le \left(\int_a^b M^{\frac{1}{p}} \left[\prod_{i=1}^{n} g_i(x)\right]^{\frac{1}{p}} dx\right)^{\frac{1}{p}} \left(\int_a^b \left[\prod_{i=1}^{n} f_i(x)\right]^{\frac{1}{q}} dx\right)^{\frac{1}{p}} \left(\int_a^b \left[\prod_{i=1}^{n} g_i(x)\right]^{\frac{1}{p}} dx\right)^{\frac{1}{q}}$$

$$\times \left(\int_a^b m^{-\frac{1}{q}} \left[\prod_{i=1}^{n} f_i(x)\right]^{\frac{1}{q}} dx\right)^{\frac{1}{q}}$$

$$= M^{\frac{1}{p^2}} \left(\int_a^b \left[\prod_{i=1}^{n} g_i(x)\right]^{\frac{1}{p}} \left[\prod_{i=1}^{n} f_i(x)\right]^{\frac{1}{q}} dx\right)^{\frac{1}{p}} m^{-\frac{1}{q^2}} \left(\int_a^b \left[\prod_{i=1}^{n} g_i(x)\right]^{\frac{1}{p}} \left[\prod_{i=1}^{n} f_i(x)\right]^{\frac{1}{q}} dx\right)^{\frac{1}{q}}$$

$$= M^{\frac{1}{p^2}} m^{-\frac{1}{q^2}} \left(\int_a^b \left[\prod_{i=1}^{n} f_i(x)\right]^{\frac{1}{q}} \left[\prod_{i=1}^{n} g_i(x)\right]^{\frac{1}{p}} dx\right)^{\frac{1}{p}} \left(\int_a^b \left[\prod_{i=1}^{n} f_i(x)\right]^{\frac{1}{q}} \left[\prod_{i=1}^{n} g_i(x)\right]^{\frac{1}{p}} dx\right)^{\frac{1}{q}}$$

$$= M^{\frac{1}{p^2}} m^{-\frac{1}{q^2}} \left(\int_a^b \left[\prod_{i=1}^{n} f_i(x)\right]^{\frac{1}{q}} \left[\prod_{i=1}^{n} g_i(x)\right]^{\frac{1}{p}} dx\right)^{\frac{1}{p}+\frac{1}{q}}$$

Since $\dfrac{1}{p}+\dfrac{1}{q}=1$, we have

$$\int_a^b \left[\prod_{i=1}^{n} f_i(x)\right]^{\frac{1}{p}} \left[\prod_{i=1}^{n} g_i(x)\right]^{\frac{1}{q}} dx \le M^{\frac{1}{p^2}} m^{-\frac{1}{q^2}} \int_a^b \left[\prod_{i=1}^{n} f_i(x)\right]^{\frac{1}{q}} \left[\prod_{i=1}^{n} g_i(x)\right]^{\frac{1}{p}} dx \qquad (19)$$

Hence, the inequality (16) is proved.

For a given positive integer p, setting $\prod_{i=1}^{n} g_i(x) \equiv 1$ in (19) gives

$$\int_a^b \left[\prod_{i=1}^{n} f_i(x)\right]^{\frac{1}{p}} dx \le \left(\int_a^b \prod_{i=1}^{n} f_i(x)\, dx\right)^{1-\frac{1}{p}}$$

Theorem 2:

Let $\prod_{i=1}^{n} f_i(x)$, $\prod_{i=1}^{n} g_i(x)$, $\prod_{i=1}^{n} t_i(x)$ and $\prod_{i=1}^{n} y_i(x)$ be non-negative functions with $0 < m \le \dfrac{\prod_{i=1}^{n} f_i(x) \prod_{i=1}^{n} t_i(x)}{\prod_{i=1}^{n} g_i(x) \prod_{i=1}^{n} y_i(x)} \le M < \infty$ on [a,b]. Then for $p > 1$ and $q > 1$ with $\dfrac{1}{p} + \dfrac{1}{q} = 1$ we have

$$\int_a^b \left[\prod_{i=1}^{n} f_i(x) \prod_{i=1}^{n} t_i(x)\right]^{\frac{1}{p}} \left[\prod_{i=1}^{n} g_i(x) \prod_{i=1}^{n} y_i(x)\right]^{\frac{1}{q}} dx \le M^{\frac{2}{p^2}} m^{-\frac{2}{q^2}}$$

$$\left(\int_a^b \prod_{i=1}^{n} f_i(x) dx\right)^{\frac{1}{q}} \left(\int_a^b \prod_{i=1}^{n} t_i(x) dx\right)^{\frac{1}{q}} \left(\int_a^b \prod_{i=1}^{n} g_i(x) dx\right)^{\frac{1}{p}} \left(\int_a^b \prod_{i=1}^{n} y_i(x) dx\right)^{\frac{1}{q}} \qquad (20)$$

Proof :

Using Holder's Inequality on the left hand side of (20), we have

$$\int_a^b \left[\prod_{i=1}^{n} f_i(x)\right]^{\frac{1}{p}} \left[\prod_{i=1}^{n} t_i(x)\right]^{\frac{1}{p}} \left[\prod_{i=1}^{n} g_i(x)\right]^{\frac{1}{q}} \left[\prod_{i=1}^{n} y_i(x)\right]^{\frac{1}{q}} dx$$

$$\le \left(\int_a^b \prod_{i=1}^{n} f_i(x) dx\right)^{\frac{1}{p}} \left(\int_a^b \prod_{i=1}^{n} t_i(x) dx\right)^{\frac{1}{p}} \left(\int_a^b \prod_{i=1}^{n} g_i(x) dx\right)^{\frac{1}{q}} \left(\int_a^b \prod_{i=1}^{n} y_i(x) dx\right)^{\frac{1}{q}} \qquad (21)$$

$$\Rightarrow \qquad \int_a^b \left[\prod_{i=1}^{n} f_i(x)\right]^{\frac{1}{p}} \left[\prod_{i=1}^{n} t_i(x)\right]^{\frac{1}{p}} \left[\prod_{i=1}^{n} g_i(x)\right]^{\frac{1}{q}} \left[\prod_{i=1}^{n} y_i(x)\right]^{\frac{1}{q}} dx$$

$$= \left(\int_a^b \left[\prod_{i=1}^{n} f_i(x)\right]^{\frac{1}{p}} \left[\prod_{i=1}^{n} f_i(x)\right]^{\frac{1}{q}} dx\right)^{\frac{1}{p}} \left(\int_a^b \left[\prod_{i=1}^{n} t_i(x)\right]^{\frac{1}{p}} \left[\prod_{i=1}^{n} t_i(x)\right]^{\frac{1}{q}} dx\right)^{\frac{1}{p}}$$

$$\times \left(\int_a^b \left[\prod_{i=1}^{n} g_i(x)\right]^{\frac{1}{p}} \left[\prod_{i=1}^{n} g_i(x)\right]^{\frac{1}{q}} dx\right)^{\frac{1}{q}} \left(\int_a^b \left[\prod_{i=1}^{n} y_i(x)\right]^{\frac{1}{p}} \left[\prod_{i=1}^{n} y_i(x)\right]^{\frac{1}{q}} dx\right)^{\frac{1}{q}}$$

Since, $\left[\prod_{i=1}^{n} f_i(x)\right]^{\frac{1}{p}} \le M^{\frac{1}{p}} \left[\prod_{i=1}^{n} g_i(x)\right]^{\frac{1}{p}}$, $\left[\prod_{i=1}^{n} t_i(x)\right]^{\frac{1}{p}} \le M^{\frac{1}{p}} \left[\prod_{i=1}^{n} y_i(x)\right]^{\frac{1}{p}}$,

$\left[\prod_{i=1}^{n} g_i(x)\right]^{\frac{1}{q}} \le m^{-\frac{1}{q}} \left[\prod_{i=1}^{n} f_i(x)\right]^{\frac{1}{q}}$ and $\left[\prod_{i=1}^{n} y_i(x)\right]^{\frac{1}{p}} \le m^{-\frac{1}{q}} \left[\prod_{i=1}^{n} t_i(x)\right]^{\frac{1}{p}}$, from the given inequality $0 < m \le \dfrac{\prod_{i=1}^{n} f_i(x) \prod_{i=1}^{n} t_i(x)}{\prod_{i=1}^{n} g_i(x) \prod_{i=1}^{n} y_i(x)} \le M < \infty$

we have,

$$\int_a^b \left[\prod_{i=1}^{n} f_i(x) \prod_{i=1}^{n} t_i(x)\right]^{\frac{1}{p}} \left[\prod_{i=1}^{n} g_i(x) \prod_{i=1}^{n} y_i(x)\right]^{\frac{1}{q}} dx$$

$$= \left(\int_a^b \left[\prod_{i=1}^{n} f_i(x)\right]^{\frac{1}{p}} dx\right)^{\frac{1}{p}} \left(\int_a^b \left[\prod_{i=1}^{n} f_i(x)\right]^{\frac{1}{q}} dx\right)^{\frac{1}{p}} \left(\int_a^b \left[\prod_{i=1}^{n} t_i(x)\right]^{\frac{1}{p}} dx\right)^{\frac{1}{p}}$$

$$\times \left(\int_a^b \left[\prod_{i=1}^{n} t_i(x)\right]^{\frac{1}{q}} dx\right)^{\frac{1}{p}} \left(\int_a^b \left[\prod_{i=1}^{n} g_i(x)\right]^{\frac{1}{p}} dx\right)^{\frac{1}{q}} \left(\int_a^b \left[\prod_{i=1}^{n} g_i(x)\right]^{\frac{1}{q}} dx\right)^{\frac{1}{q}}$$

$$\times \left(\int_a^b \left[\prod_{i=1}^{n} y_i(x)\right]^{\frac{1}{p}} dx\right)^{\frac{1}{q}} \left(\int_a^b \left[\prod_{i=1}^{n} y_i(x)\right]^{\frac{1}{q}} dx\right)^{\frac{1}{q}}$$

$$\le \left(\int_a^b M^{\frac{1}{p}} \left[\prod_{i=1}^{n} g_i(x)\right]^{\frac{1}{p}} dx\right)^{\frac{1}{p}} \left(\int_a^b \left[\prod_{i=1}^{n} f_i(x)\right]^{\frac{1}{q}} dx\right)^{\frac{1}{p}} \left(\int_a^b M^{\frac{1}{p}} \left[\prod_{i=1}^{n} y_i(x)\right]^{\frac{1}{p}} dx\right)^{\frac{1}{p}}$$

$$\times \left(\int_a^b \left[\prod_{i=1}^{n} t_i(x)\right]^{\frac{1}{q}} dx\right)^{\frac{1}{p}} \left(\int_a^b \left[\prod_{i=1}^{n} g_i(x)\right]^{\frac{1}{p}} dx\right)^{\frac{1}{q}} \left(\int_a^b m^{-\frac{1}{q}} \left[\prod_{i=1}^{n} f_i(x)\right]^{\frac{1}{q}} dx\right)^{\frac{1}{q}}$$

$$\times \left(\int_a^b \left[\prod_{i=1}^{n} y_i(x)\right]^{\frac{1}{p}} dx\right)^{\frac{1}{q}} \left(\int_a^b m^{-\frac{1}{q}} \left[\prod_{i=1}^{n} t_i(x)\right]^{\frac{1}{q}} dx\right)^{\frac{1}{q}}$$

$$= M^{\frac{1}{p^2}} \left(\int_a^b \left[\prod_{i=1}^{n} g_i(x)\right]^{\frac{1}{p}} dx\right)^{\frac{1}{p}} \left(\int_a^b \left[\prod_{i=1}^{n} f_i(x)\right]^{\frac{1}{q}} dx\right)^{\frac{1}{p}} M^{\frac{1}{p^2}} \left(\int_a^b \left[\prod_{i=1}^{n} y_i(x)\right]^{\frac{1}{p}} dx\right)^{\frac{1}{p}}$$

$$\times \left(\int_a^b \left[\prod_{i=1}^{n} t_i(x)\right]^{\frac{1}{q}} dx\right)^{\frac{1}{p}} \left(\int_a^b \left[\prod_{i=1}^{n} g_i(x)\right]^{\frac{1}{p}} dx\right)^{\frac{1}{q}} m^{-\frac{1}{q^2}} \left(\int_a^b \left[\prod_{i=1}^{n} f_i(x)\right]^{\frac{1}{q}} dx\right)^{\frac{1}{q}}$$

$$\times \left(\int_a^b \left[\prod_{i=1}^{n} y_i(x)\right]^{\frac{1}{p}} dx\right)^{\frac{1}{q}} m^{-\frac{1}{q^2}} \left(\int_a^b \left[\prod_{i=1}^{n} t_i(x)\right]^{\frac{1}{q}} dx\right)^{\frac{1}{q}}$$

$$= M^{\frac{2}{p^2}} m^{-\frac{2}{q^2}} \left(\int_a^b \left[\prod_{i=1}^n f_i(x) \right]^{\frac{1}{q}} \left[\prod_{i=1}^n g_i(x) \right]^{\frac{1}{p}} dx \right)^{\frac{1}{p}} \left(\int_a^b \left[\prod_{i=1}^n t_i(x) \right]^{\frac{1}{q}} \left[\prod_{i=1}^n y_i(x) \right]^{\frac{1}{p}} dx \right)^{\frac{1}{p}}$$

$$\times \left(\int_a^b \left[\prod_{i=1}^n f_i(x) \right]^{\frac{1}{q}} \left[\prod_{i=1}^n g_i(x) \right]^{\frac{1}{p}} dx \right)^{\frac{1}{q}} \left(\int_a^b \left[\prod_{i=1}^n t_i(x) \right]^{\frac{1}{q}} \left[\prod_{i=1}^n y_i(x) \right]^{\frac{1}{p}} dx \right)^{\frac{1}{q}}$$

$$= M^{\frac{2}{p^2}} m^{-\frac{2}{q^2}} \left(\int_a^b \left[\prod_{i=1}^n f_i(x) \right]^{\frac{1}{q}} \left[\prod_{i=1}^n g_i(x) \right]^{\frac{1}{p}} dx \right)^{\frac{1}{p}+\frac{1}{q}} \left(\int_a^b \left[\prod_{i=1}^n t_i(x) \right]^{\frac{1}{q}} \left[\prod_{i=1}^n y_i(x) \right]^{\frac{1}{p}} dx \right)^{\frac{1}{p}+\frac{1}{q}}$$

Since $\frac{1}{p}+\frac{1}{q}=1$, we have

$$\int_a^b \left[\prod_{i=1}^n f_i(x) \prod_{i=1}^n t_i(x) \right]^{\frac{1}{p}} \left[\prod_{i=1}^n g_i(x) \right]^{\frac{1}{q}} \left[\prod_{i=1}^n y_i(x) \right]^{\frac{1}{q}} dx \le M^{\frac{1}{p^4}} m^{-\frac{1}{q^4}} \int_a^b \left[\prod_{i=1}^n f_i(x) \right]^{\frac{1}{q}}$$

$$\left[\prod_{i=1}^n g_i(x) \right]^{\frac{1}{p}} \left[\prod_{i=1}^n t_i(x) \right]^{\frac{1}{q}} \left[\prod_{i=1}^n y_i(x) \right]^{\frac{1}{p}} dx \tag{22}$$

Hence, the inequality (20) is proved.

Remarks :

(1) For a given positive integer p, setting $\prod_{i=1}^n g_i(x) \equiv 1$ and $\prod_{i=1}^n y_i(x) \equiv 1$ in (22) gives

$$\int_a^b \left[\prod_{i=1}^n (f_i(x).t_i(x)) \right]^{\frac{1}{p}} dx \le \left(\int_a^b \prod_{i=1}^n [f_i(x).t_i(x)]\, dx \right)^{1-\frac{1}{p}}$$

(2) The result is valid for $2n$-terms for all $n \in N$.

(3) Since the given bounds cannot corrollate with odd functions, then the results are not valid for odd terms.

Applications

Some examples were considered as application for the generalization of QI's integral inequalities as:

For a given positive integer p, setting $\prod_{i=1}^n g_i(x) \equiv 1$ in (19) gives

$$\int_a^b \left[\prod_{i=1}^n f_i(x) \right]^{\frac{1}{p}} dx \le \left(\int_a^b \prod_{i=1}^n f_i(x)\, dx \right)^{1-\frac{1}{p}}$$

Problem 1. Let f_1 and f_2 be nonnegative functions with $0 < m \le \frac{f_1(x)}{f_2(x)} \le M < \infty$ on $[a,b]$, for $p > 1$. Suppose $f_2 = f_1^{'}$, $p = 3$ and it is integrable over $[a,b]$, then the inequality holds:

$$\int_a^b \left(f_1(x)\ f_2(x)\, dx\right)^{\frac{1}{p}} \le \left(\int_a^b f_1(x) f_1^{'}(x)\, dx \right)^{1-\frac{1}{p}}$$

Setting $p=3$, $f_1(x)=\exp^{3x}$, $f_2(x)=3\exp^{3x}$, on $[1,2]$

$$\int_1^2 \left(\exp^{3x}.3\exp^{3x}\right)^{\frac{1}{3}} dx \le \left(\int_1^2 \exp^{3x}.3\exp^{3x}\, dx \right)^{1-\frac{1}{3}}$$

$$\Rightarrow \int_1^2 \left(3\exp^{6x}\right)^{\frac{1}{3}} dx \le \left(\int_1^2 3\exp^{6x}\, dx \right)^{\frac{2}{3}}$$

$$\Rightarrow \frac{3^{\frac{1}{3}}}{2}\left[\exp^{2x}\right]_1^2 \le \left(\frac{\exp^{12}-\exp^{6}}{2} \right)^{\frac{2}{3}}$$

$$\Rightarrow \quad 34.04 < 1874.78$$

Problem 2. Let f_1 and f_2 be nonnegative functions with $0 < m \le \frac{f_1(x)}{f_2(x)} \le M < \infty$ on $[a,b]$, for $p>1$. Suppose $f_1 = \exp^{\frac{x}{2}}$, $f_2 = \exp^{3x}$, $p=4$ and it is integrable over $[1,2]$, then the inequality below holds:

$$\int_1^2 \left(\exp^{\frac{x}{2}}.\exp^{3x} \right)^{\frac{1}{4}} dx \le \left(\int_1^2 \exp^{\frac{x}{2}}.\exp^{3x}\, dx \right)^{1-\frac{1}{4}}$$

$$\Rightarrow \frac{8}{7}\left[\exp^{\frac{7x}{8}} \right]_1^2 \le \left[\frac{2}{7}(\exp^{7}-\exp^{\frac{7}{2}}) \right]^{\frac{3}{4}}$$

$$\Rightarrow \frac{8}{7}\left[3.3557\right] \le \left(303.861285\ 7)\right)^{\frac{3}{4}}$$

Therefore,

$$\Rightarrow 3.8351 < 72.7791$$

Problem 3. Suppose f_1, f_2, and f_3 are nonnegative functions with $0 < m \le \frac{f_1(x)}{f_2(x)} \le M < \infty$ on $[a,b]$, for $p>1$. Suppose $f_1 = \exp^{5x+2}$, $f_2 = \exp^{2x}$, $f_3 = \exp^{3x+1}$, $p=2$ and it is integrable over $[1,2]$, then the inequality below holds:

$$\int_1^2 \left(\exp^{5x+2}.\exp^{2x}.\exp^{3x+1}\right)^{\frac{1}{2}} dx \le \left(\int_1^2 \exp^{5x+2}.\exp^{2x}.\exp^{3x+1} dx \right)^{1-\frac{1}{2}}$$

$$\Rightarrow \frac{1}{5}\left[\exp^{10+\frac{3}{2}} - \exp^{5+\frac{3}{2}} \right] \le \left[\frac{1}{10}(\exp^{20+3}-\exp^{10+3}) \right]^{\frac{1}{2}}$$

$$\Rightarrow 19610.1259 < 31215.9591$$

Problem 4. Suppose f_1, f_2, f_3 and f_4 are nonnegative functions with $0 < m \le \frac{f_1(x)}{f_2(x)} \le M < \infty$ on

$[a,b]$, for $p>1$. Suppose $f_1=\exp^{3x}$, $f_2=\frac{\exp^{x}}{2}$, $f_3=\exp^{x}$, and $f_4=\exp^{\frac{x}{2}}$, $p=5$ and it is integrable over $[3,4]$, then the inequality below holds:

$$\int_3^4\left(\exp^{3x}.\frac{\exp^{x}}{2}.\exp^{x}.\exp^{\frac{x}{2}}\right)^{\frac{1}{5}}dx\le\left(\int_3^4\exp^{3x}.\frac{\exp^{x}}{2}.\exp^{x}.\exp^{\frac{x}{2}}\,dx\right)^{1-\frac{1}{5}}$$

$$\Rightarrow 0.7914\left(\exp^{4.4}-\exp^{3.3}\right)\le\left(\frac{1}{11}(\exp^{22}-\exp^{16.5})\right)^{\frac{4}{5}}$$

$$\Rightarrow 43.0033 < 6442374.01\ 28$$

Problem 5. Suppose f_1, f_2, f_3 f_4, and f_5 are nonnegative functions with $0<m\le\frac{f_1(x)}{f_2(x)}\le M<\infty$ on $[a,b]$, for $p>1$. Suppose $f_1=\exp^{3x-5}$, $f_2=\exp^{3-5x}$, $f_3=\exp^{x}$, $f_4=5\exp^{2x}$, and $f_5=\exp^{4}$, $p=6$ and it is integrable over $[2,3]$, then the inequality below holds:

$$\int_2^3\left(\exp^{3x-5}.\exp^{3-5x}.\exp^{x}5\exp^{2x}.\exp^{4}\right)^{\frac{1}{6}}dx\le\left(\int_2^3\exp^{3x-5}.\exp^{3-5x}.\exp^{x}.5\exp^{2x}.\exp^{4}\,dx\right)^{1-\frac{1}{6}}$$

$$\Rightarrow 5^{\frac{1}{6}}.6\left[\exp^{\frac{5}{6}}-\exp^{\frac{2}{3}}\right]\le\left(5[148.4132-54.5982]\right)^{\frac{5}{6}}$$

$$\Rightarrow 2.772<168.2811$$

CONCLUSION

This paper has investigated, extended and generalized some results on QI type integral inequality involving non-negative and continuous functions. Some applications were also considered. However, there may be need to extend this work in the future, by considering trigonometric functions with the aim of obtaining sharp bound, since trigonometric functions are not strictly non-negative (not strictly positive) and it is believed that useful results can be derived from the idea.

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